a lowkey intro to synthetic differential geometry

Daniel Cicala

19 May 2018

ucr

"The reason why I have postponed for so long these investigations, which are basic to my other work in this field, is essentially the following. I found these theories originally by **synthetic** *considerations. But I soon realized that, as expedient the* **synthetic** *method is for discovery, as difficult it is to give a clear exposition on* **synthetic** *investigations, which deal with objects that till now have almost exclusively been considered* **analytically**. *After long vacillations, I have decided to use a half* **synthetic**, *half* **analytic** *form. I hope my work will serve to bring justification to the* **synthetic** *method besides the* **analytical** *one."*

–**Sophus Lie**, Allgemeine Theorie der partiellen Differentialgleichungen erster Ordnung, Math. Ann. 9 (1876).

tldr; Sophus Lie worked synthetically (via infinitesimals) but communicated analytically due to a lack of synthetic language.

"The reason why I have postponed for so long these investigations, which are basic to my other work in this field, is essentially the following. I found these theories originally by **synthetic** *considerations. But I soon realized that, as expedient the* **synthetic** *method is for discovery, as difficult it is to give a clear exposition on* **synthetic** *investigations, which deal with objects that till now have almost exclusively been considered* **analytically**. *After long vacillations, I have decided to use a half* **synthetic**, *half* **analytic** *form. I hope my work will serve to bring justification to the* **synthetic** *method besides the* **analytical** *one."*

–**Sophus Lie**, Allgemeine Theorie der partiellen Differentialgleichungen erster Ordnung, Math. Ann. 9 (1876).

tldr; Sophus Lie worked synthetically (via infinitesimals) but communicated analytically due to a lack of synthetic language.

analytical vs. synthetic

analytical mathematics: build things from set theory

synthetic mathematics: find models of axiomatic systems from existing systems.

building our axioms

an infinitesimal injection

R will be our "reals" and D our "infinitesimals"

axiom v1: Let $D \coloneqq \{d \in R : d^2 = 0\}$. Then any function $g\colon D \to R$ is of the form

$$g(d) = a + bd$$

for some unique $a, b \in R$.

Given $f\colon R \to R$, *define* its derivative at $x \in R$ as follows:

Let $g(d) \coloneqq f(x + d)$.

By **axiom v1**, $g(d) = g(0) + bd = f(x) + bd$. Take

$$f'(x) \coloneqq b$$

an infinitesimal injection

R will be our "reals" and D our "infinitesimals"

axiom v1: Let $D \coloneqq \{d \in R : d^2 = 0\}$. Then any function $g\colon D \to R$ is of the form

$$g(d) = a + bd$$

for some unique $a, b \in R$.

Given $f\colon R \to R$, *define* its derivative at $x \in R$ as follows:

Let $g(d) \coloneqq f(x + d)$.

By **axiom v1**, $g(d) = g(0) + bd = f(x) + bd$. Take

$$f'(x) \coloneqq b$$

an infinitesimal injection

R will be our "reals" and D our "infinitesimals"

axiom v1: Let $D := \{d \in R : d^2 = 0\}$. Then any function $g\colon D \to R$ is of the form

$$g(d) = a + bd$$

for some unique $a, b \in R$.

Given $f\colon R \to R$, *define* its derivative at $x \in R$ as follows:

Let $g(d) := f(x + d)$.

By **axiom v1**, $g(d) = g(0) + bd = f(x) + bd$. Take

$$f'(x) := b$$

an infinitesimal injection

R will be our "reals" and D our "infinitesimals"

axiom v1: Let $D := \{d \in R : d^2 = 0\}$. Then any function $g \colon D \to R$ is of the form

$$g(d) = a + bd$$

for some unique $a, b \in R$.

Given $f \colon R \to R$, *define* its derivative at $x \in R$ as follows:

Let $g(d) := f(x + d)$.

By **axiom v1**, $g(d) = g(0) + bd = f(x) + bd$. Take

$$f'(x) := b$$

what have we done?!!?!?

Every function $R \to R$ has a derivative!

Is this a problem?

Theorem: Assuming axiom v1: $R = \{0\}$

the proof contains a line "for $d \in D$, either $d = 0$ or $d \neq 0$."

what have we done?!!?!?

Every function $R \to R$ has a derivative!

Is this a problem?

Theorem: Assuming axiom v1: $R = \{0\}$

the proof contains a line "for $d \in D$, either $d = 0$ or $d \neq 0$."

what have we done?!!?!?

Every function $R \to R$ has a derivative!

Is this a problem?

Theorem: Assuming axiom v1: $R = \{0\}$

the proof contains a line "for $d \in D$, either $d = 0$ or $d \neq 0$."

run away!!! run away!!!

Do we...

change our axiom? NO

weaken our logic? YES

say goodbye to classical logic

say hello to intuitionistic logic

intuitionistic logic := classical logic - $(p \vee \neg p)$

run away!!! run away!!!

Do we...

change our axiom? NO

weaken our logic? YES

say goodbye to classical logic

say hello to intuitionistic logic

intuitionistic logic := classical logic - $(p \vee \neg p)$

run away!!! run away!!!

Do we...

change our axiom? NO

weaken our logic? YES

say goodbye to classical logic

say hello to intuitionistic logic

intuitionistic logic := classical logic - $(p \vee \neg p)$

run away!!! run away!!!

Do we...

change our axiom? NO

weaken our logic? YES

say goodbye to classical logic

say hello to intuitionistic logic

intuitionistic logic := classical logic - $(p \vee \neg p)$

in the bargaining stage of grief

In short, we trade LEM for our axiom.

Now, $\neg(R = \{0\})$ because we cannot use the statement "for $d \in D$, either $d = 0$ or $d \neq 0$".

In fact, **axiom v1 – LEM** gives R is a field.

in the bargaining stage of grief

In short, we trade LEM for our axiom.

Now, $\neg(R = \{0\})$ because we cannot use the statement "for $d \in D$, either $d = 0$ or $d \neq 0$".

In fact, **axiom v1 – LEM** gives R is a field.

in the bargaining stage of grief

In short, we trade LEM for our axiom.

Now, $\neg(R = \{0\})$ because we cannot use the statement "for $d \in D$, either $d = 0$ or $d \neq 0$".

In fact, **axiom v1 – LEM** gives R is a field.

a field with nilpotents

For R to be a field, we need

$$(x \neq 0) \quad \Rightarrow \quad x \text{ is invertible}$$

or in the contrapositive

$$x \text{ is not invertible} \quad \Rightarrow \quad \neg(x \neq 0)$$

Infintesimals d with $d^2 = 0$ are not invertible, so

$$d \text{ is not non-zero}$$

a field with nilpotents

For R to be a field, we need

$$(x \neq 0) \quad \Rightarrow \quad x \text{ is invertible}$$

or in the contrapositive

$$x \text{ is not invertible} \quad \Rightarrow \quad \neg(x \neq 0)$$

Infintesimals d with $d^2 = 0$ are not invertible, so

$$d \text{ is not non-zero}$$

neither the thing or not the thing

If d is NOT non-zero, is $d = 0$?

theorem: $D \neq \{0\}$.

Competing facts:

- $\exists d \in D,\ d \neq 0$
- $\forall d \in D,\ \neg(d \neq 0)$

With LEM out of our way, no problem.

neither the thing or not the thing

If d is NOT non-zero, is $d = 0$?

theorem: $D \neq \{0\}$.

Competing facts:

- $\exists d \in D,\ d \neq 0$
- $\forall d \in D,\ \neg(d \neq 0)$

With LEM out of our way, no problem.

neither the thing or not the thing

If d is NOT non-zero, is $d = 0$?

theorem: $D \neq \{0\}$.

Competing facts:

- $\exists d \in D$, $d \neq 0$
- $\forall d \in D$, $\neg(d \neq 0)$

With LEM out of our way, no problem.

jumping ahead

Nilsquares aren't subtle enough to work in multivariable settings.

define: An *infinitesimal object* is a subset $C \subseteq R^n$ defined by polynomial equations such that, for any $(c_i) \in C$, each c_i is nilpotent.

example: $(d_1, d_2, d_3, d_4) \in R^4$ such that

$$d_1^3 = d_2^4 = d_3^2 = d_4^2 = 0 \quad , \quad d_1 + d_1 d_2 + d^3 = 0$$

final axiom: If C is an infinitesimal object, every function $f : C \to R$ is uniquely determined by a polynomial or power series in C .

jumping ahead

Nilsquares aren't subtle enough to work in multivariable settings.

define: An *infinitesimal object* is a subset $C \subseteq R^n$ defined by polynomial equations such that, for any $(c_i) \in C$, each c_i is nilpotent.

example: $(d_1, d_2, d_3, d_4) \in R^4$ such that

$$d_1^3 = d_2^4 = d_3^2 = d_4^2 = 0 \quad , \quad d_1 + d_1 d_2 + d^3 = 0$$

final axiom: If C is an infinitesimal object, every function $f : C \to R$ is uniquely determined by a polynomial or power series in C .

jumping ahead

Nilsquares aren't subtle enough to work in multivariable settings.

define: An *infinitesimal object* is a subset $C \subseteq R^n$ defined by polynomial equations such that, for any $(c_i) \in C$, each c_i is nilpotent.

example: $(d_1, d_2, d_3, d_4) \in R^4$ such that

$$d_1^3 = d_2^4 = d_3^2 = d_4^2 = 0 \quad , \quad d_1 + d_1 d_2 + d^3 = 0$$

final axiom: If C is an infinitesimal object, every function $f : C \to R$ is uniquely determined by a polynomial or power series in C .

jumping ahead

Nilsquares aren't subtle enough to work in multivariable settings.

define: An *infinitesimal object* is a subset $C \subseteq R^n$ defined by polynomial equations such that, for any $(c_i) \in C$, each c_i is nilpotent.

example: $(d_1, d_2, d_3, d_4) \in R^4$ such that

$$d_1^3 = d_2^4 = d_3^2 = d_4^2 = 0 \quad , \quad d_1 + d_1 d_2 + d^3 = 0$$

final axiom: If C is an infinitesimal object, every function $f : C \to R$ is uniquely determined by a polynomial or power series in C .

smoothness

microlinearity "=" smooth structure

Categorical notions determine if a set is "smooth"

(definition) A set M is called *microlinear* if any diagram of infinitesimal objects perceived by R as a colimit is also perceived by M as a colimit.

Translation: "all local infinitesimal structure possessed by R is possessed by microlinear sets"

(examples) R, R^n , products of microlinear, limits of microlinear (e.g. zero-sets of type $R^n \to R$), M^X for M microlinear and X any set.

microlinearity "=" smooth structure

Categorical notions determine if a set is "smooth"

(definition) A set M is called *microlinear* if any diagram of infinitesimal objects perceived by R as a colimit is also perceived by M as a colimit.

Translation: "all local infinitesimal structure possessed by R is possessed by microlinear sets"

(examples) R, R^n , products of microlinear, limits of microlinear (e.g. zero-sets of type $R^n \to R$), M^X for M microlinear and X any set.

microlinearity "=" smooth structure

Categorical notions determine if a set is "smooth"

(definition) A set M is called *microlinear* if any diagram of infinitesimal objects perceived by R as a colimit is also perceived by M as a colimit.

Translation: "all local infinitesimal structure possessed by R is possessed by microlinear sets"

(examples) R, R^n , products of microlinear, limits of microlinear (e.g. zero-sets of type $R^n \to R$), M^X for M microlinear and X any set.

microlinearity "=" smooth structure

Categorical notions determine if a set is "smooth"

(definition) A set M is called *microlinear* if any diagram of infinitesimal objects perceived by R as a colimit is also perceived by M as a colimit.

Translation: "all local infinitesimal structure possessed by R is possessed by microlinear sets"

(examples) R, R^n , products of microlinear, limits of microlinear (e.g. zero-sets of type $R^n \to R$), M^X for M microlinear and X any set.

differential geometry basics

(definition) Let M be microlinear and $m \in M$. A *tangent vector* to M at m is any map $t: D \to M$ with $t(0) = m$.

(prop) $T_m M$ is a vector space.

(definition) Let $f: M \to N$ be a map between microlinear sets. The *differential* of f at m is $d_m f: T_m M \to T_{fm} N$ given by

$$(d_m f(t))(d) = f(t(d))$$

(prop) $d_m f$ is a linear map.

(definition) For M microlinear, its *tangent bundle* is $TM := M^D$.

finally, differential geometry

(definition) Let M be microlinear and $m \in M$. A *tangent vector* to M at m is any map $t\colon D \to M$ with $t(0) = m$.

(prop) $T_m M$ is a vector space.

(definition) Let $f\colon M \to N$ be a map between microlinear sets. The *differential* of f at m is $d_m f\colon T_m M \to T_{fm} N$ given by

$$(d_m f(t))(d) = f(t(d))$$

(prop) $d_m f$ is a linear map.

(definition) For M microlinear, its *tangent bundle* is $TM := M^D$.

finally, differential geometry

(definition) Let M be microlinear and $m \in M$. A *tangent vector* to M at m is any map $t\colon D \to M$ with $t(0) = m$.

(prop) $T_m M$ is a vector space.

(definition) Let $f\colon M \to N$ be a map between microlinear sets. The *differential* of f at m is $d_m f\colon T_m M \to T_{fm} N$ given by

$$(d_m f(t))(d) = f(t(d))$$

(prop) $d_m f$ is a linear map.

(definition) For M microlinear, its *tangent bundle* is $TM := M^D$.

finally, differential geometry

(definition) Let M be microlinear and $m \in M$. A *tangent vector* to M at m is any map $t\colon D \to M$ with $t(0) = m$.

(prop) $T_m M$ is a vector space.

(definition) Let $f\colon M \to N$ be a map between microlinear sets. The *differential* of f at m is $d_m f\colon T_m M \to T_{fm} N$ given by

$$(d_m f(t))(d) = f(t(d))$$

(prop) $d_m f$ is a linear map.

(definition) For M microlinear, its *tangent bundle* is $TM := M^D$.

finally, differential geometry

(definition) Let M be microlinear and $m \in M$. A *tangent vector* to M at m is any map $t \colon D \to M$ with $t(0) = m$.

(prop) $T_m M$ is a vector space.

(definition) Let $f \colon M \to N$ be a map between microlinear sets. The *differential* of f at m is $d_m f \colon T_m M \to T_{fm} N$ given by

$$(d_m f(t))(d) = f(t(d))$$

(prop) $d_m f$ is a linear map.

(definition) For M microlinear, its *tangent bundle* is $TM := M^D$.

Lie groups

(definition) A *Lie group* is a microlinear group

This includes classical Lie groups and more.

(definition) A Lie algebra of a Lie group G is $T_e G$.

(bracket) Let $X, Y \in T_e G$, that is $X, Y\colon D \to G$. Define $X * Y\colon D \times D \to G$ by

$$(d, e) \mapsto X(d)Y(e)X(-d)Y(-e)$$

By microlinearity, this determines a map

$$[X, Y]\colon D \to G$$

satisfying bracket equations.

Lie groups

(definition) A *Lie group* is a microlinear group

This includes classical Lie groups and more.

(definition) A Lie algebra of a Lie group G is $T_e G$.

(bracket) Let $X, Y \in T_e G$, that is $X, Y \colon D \to G$. Define $X * Y \colon D \times D \to G$ by

$$(d, e) \mapsto X(d)Y(e)X(-d)Y(-e)$$

By microlinearity, this determines a map

$$[X, Y] \colon D \to G$$

satisfying bracket equations.

Lie groups

(definition) A *Lie group* is a microlinear group

This includes classical Lie groups and more.

(definition) A Lie algebra of a Lie group G is $T_e G$.

(bracket) Let $X, Y \in T_e G$, that is $X, Y \colon D \to G$. Define $X * Y \colon D \times D \to G$ by

$$(d, e) \mapsto X(d)Y(e)X(-d)Y(-e)$$

By microlinearity, this determines a map

$$[X, Y] \colon D \to G$$

satisfying bracket equations.

vector fields

(definition) A vector field on M is a section of $TM := M^D$.

But we have

$$\frac{\frac{M \to M^D}{M \times D \to M}}{D \to M^M}$$

So a vector field on M IS a tangent vector on the microlinear set M^M.

(theorem) The set of vector fields on M coincides with $T_{id}(M^M)$

vector fields

(definition) A vector field on M is a section of $TM := M^D$.

But we have

$$\frac{\dfrac{M \to M^D}{M \times D \to M}}{D \to M^M}$$

So a vector field on M IS a tangent vector on the microlinear set M^M.

(theorem) The set of vector fields on M coincides with $T_{id}(M^M)$

vector fields

(definition) A vector field on M is a section of $TM := M^D$.

But we have

$$\cfrac{\cfrac{M \to M^D}{M \times D \to M}}{D \to M^M}$$

So a vector field on M IS a tangent vector on the microlinear set M^M.

(theorem) The set of vector fields on M coincides with $T_{id}(M^M)$

a model of synthetic differential geometry

show me a sdg

Everything so far follows straight from axioms. But axioms are cheap without models.

We will construct a model using "set theory".

Given a sufficiently nice category (i.e. a topos), we

- take objects as "sets"
- arrows are "functions"
- (co)limits as logical operations, e.g. $\{x|f(x) = g(x)\}$ is an equalizer.

show me a sdg

Everything so far follows straight from axioms. But axioms are cheap without models.

We will construct a model using "set theory".

Given a sufficiently nice category (i.e. a topos), we

take objects as "sets"
arrows are "functions"
(co)limits as logical operations, e.g. $\{x|f(x) = g(x)\}$ is an equalizer.

Let's build a "set theory" for sdg.

(definition) A C^∞ -ring is a ring A equipped with, for every smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$, a map $\hat{f}\colon A^n \to A$ satisfying some conditions. This gives a category $C^\infty\mathbf{Rng}$

$C^\infty\mathbf{Rng}^{\mathrm{op}}$ includes smooth manifolds via an embedding

$$M \mapsto \text{ smooth functions of type } M \to \mathbb{R},$$

$$(f\colon M \to N) \mapsto (f^*\colon \{N \to \mathbb{R}\} \to \{M \to \mathbb{R}\})$$

$C^\infty\mathbf{Rng}^{\mathrm{op}}$ includes infinitesimal objects:

$$\mathbb{R}[\varepsilon] \coloneqq \mathbb{R}[x]/(x^2)$$

Let's build a "set theory" for sdg.

(definition) A C^∞ -ring is a ring A equipped with, for every smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$, a map $\hat{f}\colon A^n \to A$ satisfying some conditions. This gives a category $C^\infty\mathbf{Rng}$

$C^\infty\mathbf{Rng}^{\mathrm{op}}$ includes smooth manifolds via an embedding

$$M \mapsto \text{ smooth functions of type } M \to \mathbb{R},$$

$$(f\colon M \to N) \mapsto (f^*\colon \{N \to \mathbb{R}\} \to \{M \to \mathbb{R}\})$$

$C^\infty\mathbf{Rng}^{\mathrm{op}}$ includes infinitesimal objects:

$$\mathbb{R}[\varepsilon] := \mathbb{R}[x]/(x^2)$$

Let's build a "set theory" for sdg.

(definition) A C^∞ -ring is a ring A equipped with, for every smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$, a map $\hat{f}\colon A^n \to A$ satisfying some conditions. This gives a category $C^\infty\mathbf{Rng}$

$C^\infty\mathbf{Rng}^{\mathrm{op}}$ includes smooth manifolds via an embedding

$$M \mapsto \text{ smooth functions of type } M \to \mathbb{R},$$

$$(f\colon M \to N) \mapsto (f^*\colon \{N \to \mathbb{R}\} \to \{M \to \mathbb{R}\})$$

$C^\infty\mathbf{Rng}^{\mathrm{op}}$ includes infinitesimal objects:

$$\mathbb{R}[\varepsilon] := \mathbb{R}[x]/(x^2)$$

Let's build a "set theory" for sdg.

(definition) A C^∞ -ring is a ring A equipped with, for every smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$, a map $\hat{f}\colon A^n \to A$ satisfying some conditions. This gives a category $C^\infty\mathbf{Rng}$

$C^\infty\mathbf{Rng}^{\mathrm{op}}$ includes smooth manifolds via an embedding

$$M \mapsto \text{ smooth functions of type } M \to \mathbb{R},$$

$$(f\colon M \to N) \mapsto (f^*\colon \{N \to \mathbb{R}\} \to \{M \to \mathbb{R}\})$$

$C^\infty\mathbf{Rng}^{\mathrm{op}}$ includes infinitesimal objects:

$$\mathbb{R}[\varepsilon] \coloneqq \mathbb{R}[x]/(x^2)$$

make it nice

$C^\infty\mathbf{Rng}^{op}$ isn't nice enough yet.

equip it with a suitable Grothendieck topology (a generalization of open sets to a category)

The category of sheaves $C^\infty\mathbf{Rng}^{op} \to \mathbf{Set}$ is a sufficiently nice category in which our axiom is true.

That is, synthetic differential geometry is real

(whatever that means).

make it nice

$C^\infty\mathbf{Rng}^{\text{op}}$ isn't nice enough yet.

equip it with a suitable Grothendieck topology (a generalization of open sets to a category)

The category of sheaves $C^\infty\mathbf{Rng}^{\text{op}} \to \mathbf{Set}$ is a sufficiently nice category in which our axiom is true.

That is, synthetic differential geometry is real

(whatever that means).

make it nice

$C^\infty\mathbf{Rng}^{\mathrm{op}}$ isn't nice enough yet.

equip it with a suitable Grothendieck topology (a generalization of open sets to a category)

The category of sheaves $C^\infty\mathbf{Rng}^{\mathrm{op}} \to \mathbf{Set}$ is a sufficiently nice category in which our axiom is true.

That is, synthetic differential geometry is real

(whatever that means).

thanks